Random planar curves
Schramm-Loewner Evolution
and Conformal Field Theory

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Introduction - lattice models in two dimensions and random planar curves

The Ising model

- configurations \( \{ s(r) = \pm 1 \} \) with \( r \in \delta^2 \mathbb{Z}^2 \cap D \)
- weighted by

\[
\exp \left( \sum_{\text{edges } rr'} \delta(s(r), s(r')) / T \right)
\]

- defines a discrete probability measure on configurations
typical configurations:

$T > T_c$

$T = T_c$

$T < T_c$
a larger domain

\[ T = T_c, \ -/+ \text{ boundary conditions} \]
cartoon version

► nested non-intersecting curves, weighted by $e^{-\text{total length}/T}$
► generalisation: weight each configuration by

$$e^{-\text{total length}/T \cdot n \text{ number of loops}}$$

► we are interested in the scaling limit: lattice spacing $\delta \to 0$ with $D$ fixed, where many properties are supposed to be universal: different values of $n$ have different scaling limits (universality classes)
Main approaches

▶ **Integrability**: find lattice models which are ‘exactly solvable’: however usually only extensive thermodynamic properties are calculable, and it is unclear which of these are universal

▶ **Conformal Field Theory**: directly describes the scaling limits of local correlation functions, eg

\[ \mathbb{E}[s(r_1)s(r_2)] - \mathbb{E}[s(r_1)] \mathbb{E}[s(r_2)]; \]
\[ \Pr(r_1 \text{ and } r_2 \text{ lie within } \epsilon \text{ of the same curve}); \]

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- these three approaches appear to be mathematically linked through the existence of **holomorphic observables**.
choose \( D = \) upper half-plane \( \mathbb{H} \), and boundary conditions such that there is a curve \( \gamma \) from 0 to \( \infty \)

in the lattice this can be constructed via an *exploration process*

let \( \gamma_t \) be the curve up to ‘time’ \( t \), and \( \overline{\gamma}_t = \gamma \backslash \gamma_t \)

conditional law of \( \overline{\gamma}_t \) given \( \gamma_t \) in \( \mathbb{H} = \) law of \( \overline{\gamma}_t \) in \( \mathbb{H} \backslash \gamma_t \)

SLE assumes that this ‘domain Markov property’ extends to the measure on the curve in the scaling limit
Loewner’s Equation

Consider the conformal mapping $g_t : \mathbb{H} \setminus \gamma_t \to \mathbb{H}$ such that

$$g_t(z) \xrightarrow{z \to \infty} z + \frac{2t}{z} + o(z^{-1})$$

so that $g_t(\text{growing tip}) = a_t$. Then

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a_t} \quad \text{(Loewner)}$$

If $\{\gamma_t\}$ is an increasing set of (random) curves which grow only at the tip then $a_t$ is a continuous (random) process on the real line.
Conformal Invariance of the Measure

Theorem (Schramm). If both the domain Markov property and conformal invariance hold then $a_t$ is proportional to standard Brownian motion: $a_t = \sqrt{\kappa} B_t$

- different values of $\kappa$ correspond to different universality classes: conjecturally $n = -2 \cos(4\pi/\kappa)$
- identifying suitable martingales of this process allows the computation of many physically interesting quantities, e.g. fractal dimension of $\gamma$: $d_f = 1 + \frac{1}{8} \kappa$
CFT

- CFT is a special case of a QFT: a collection of functions $D^N \setminus \{\text{coincident points}\} \to \mathbb{C}$, denoted by $\langle \phi_1(r_1) \ldots \phi_N(r_N) \rangle_D$
- they satisfy certain axioms, e.g. the short-distance expansion
  $$\phi_i(r_i) \cdot \phi_j(r_j) = \sum_k C_{ijk}(r_i - r_k) \phi_k(\frac{1}{2}(r_i + r_j))$$
- the conjectured scaling limit of correlation functions of lattice models gives an example of a QFT:
  $$\lim_{\delta \to 0} \delta^{-x_1 \cdots -x_N} \mathbb{E}\left[\phi_1^{\text{lat}}(r_1) \ldots \phi_N^{\text{lat}}(r_N)\right] = \langle \phi_1(r_1) \ldots \phi_N(r_N) \rangle$$
- at $T = T_c$ the QFT is massless and the absence of any intrinsic scale implies scale covariance
  $$\langle \phi_1(r_1) \ldots \phi_N(r_N) \rangle_D = \lambda^{x_1 \cdots +x_N} \langle \phi_1(\lambda r_1) \ldots \phi_N(\lambda r_N) \rangle_{\lambda D}$$
  written more compactly as
  $$\phi_j(r) = \lambda^{x_j} \phi_j(\lambda r)$$
in a CFT this gets promoted to *conformal covariance*: under a conformal mapping $f : z \rightarrow z'$

$$\phi_j(z, \bar{z}) = f'(z)^{h_j} (f'(z_j)^*)^{\bar{h}_j} \phi_j(z', \bar{z}')$$

where $h_j + \bar{h}_j = x_j$ is the *dimension* and $h_j - \bar{h}_j = \sigma_j$ is the *conformal spin* of $\phi_j$.

a special role is played by *holomorphic* fields $\phi_\sigma(z)$ such that $(h, \bar{h}) = (\sigma, 0)$ (and similarly antiholomorphic fields)

the $W_n$ generate an infinite dimensional algebra $\mathcal{W}$ and the fields $\Phi_j$ of the CFT fall into highest weight representations of $\mathcal{W} \otimes \overline{\mathcal{W}}$

this allows a classification and characterisation of CFTs
Holomorphic fields and SLE

we can associate a field of conformal spin \( \sigma \) with the curve \( \gamma \):

\[
\phi_\sigma(z, \bar{z}) = 1_{\{z\in\gamma\}} e^{-i\sigma \theta_{0z}(\gamma)}
\]

where \( \theta_{0z}(\gamma) = \) winding angle of \( \gamma \) between 0 and \( z \). Note that as \( z \to \partial D \), \( \arg \phi_\sigma \) is fixed. If \( \phi_\sigma \) is holomorphic \( \langle \phi_\sigma(z) \rangle_{\mathbb{H}} \propto z^{-\sigma} \), and

\[
\langle \phi_\sigma(z) \rangle_{\mathbb{H}} = \mathbb{E}_{\gamma_t} \left[ \langle \phi_\sigma(z) \rangle_{\mathbb{H}\setminus\gamma_t} \right] = \mathbb{E}_{g_t} \left[ g_t'(z)^\sigma \langle \phi_\sigma(g_t(z)) \rangle_{\mathbb{H}} \right]
\]

\[
= \mathbb{E}_{a_t} \left[ \frac{g_t'(z)^\sigma}{(g_t(z) - a_t)^\sigma} \right]
\]

[\cdots] is a martingale: evaluating this for large \( z \) gives \( \mathbb{E}[a_t] = 0 \) and \( \mathbb{E}[a_t^2] = \kappa t \) where \( \kappa = 8/(\sigma + 1) \) \( \Rightarrow \gamma \) is SLE
Discrete holomorphicity

- at the level of lattice models we can often show directly from the local weights that, for suitable $\sigma$, $\phi_{\sigma}(z)$ is discretely holomorphic, e.g.

\[ \phi_{\sigma}(z_1) + \omega \phi_{\sigma}(z_2) + \omega^2 \phi_{\sigma}(z_3) = 0 \]
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However, it is a highly non-trivial step to show that this implies that, in the scaling limit \( \delta \to 0 \), \( \phi_\sigma(z) \) converges to a truly holomorphic field, in fact only in one case has this been carried out completely:

**Theorem (Smirnov):** For the Ising model on the square and triangular lattices \( \phi_\frac{1}{2}(z) \) defined in this way converges to a holomorphic field obeying the correct boundary conditions, and hence the scaling limit of the lattice curve \( \gamma \) is SLE with \( \kappa = 3 \).

**Remark:** in this case \( \phi_\frac{1}{2} \) is related to the well-known Ising fermion in other exact solutions of the model.
by now, discretely holomorphic observables have been identified in many lattice models.

the values of $\sigma$ correspond to those of the holomorphic parafermionic fields in the conjectured corresponding CFT

discrete holomorphicity imposes linear relations between the local weights of the lattice models – in all known cases these turn out to be the critical subspace of the integrable manifold in the space of all possible weights - *i.e.* they satisfy the Yang-Baxter equations.
Summary

- Schramm-Loewner Evolution and Conformal Field Theory are complementary ways of describing the conjectured scaling limit of random curves in critical lattice models.
- The inputs of SLE are fewer and more easily susceptible to verification - e.g. by proving the existence of suitable holomorphic observables.
- In some cases it is possible to derive many of the fundamental equations of CFT from SLE (e.g. the conformal Ward identities).
- Holomorphic observables play an important role in:
  - Showing that the scaling limit of lattice curves is SLE.
  - Identifying the holomorphic fields which are the building blocks of the corresponding CFT.
  - Identifying which critical lattice models are also integrable.
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  - Identifying the holomorphic fields which are the building blocks of the corresponding CFT.
  - Identifying which critical lattice models are also integrable.
- Many unresolved questions - why integrability? Extension to Conformal Loop Ensemble (CLE)? . . .