Subfactors and Planar Algebras

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Quantum Mechanics

Schroedinger - wave mechanics
Heisenberg - matrix mechanics

united with the theory of operators on **Hilbert space** - infinite dimensional complex inner product space

States: vectors (rays) in a Hilbert space
Obeservables: self-adjoint operators
Measurements: eigenvalues

von Neumann algebras

Murray-von Neumann: "Rings of Operators"

Double Commutant Theorem: Let M be a self-adjoint algebra of bounded operators on a complex Hilbert space containing the identity, and let M' be its commutant. TFAE:

(1) $M = M''$

(2) M is closed in the topology of pointwise convergence

Definition

A **factor** is a von Neumann algebras with trivial center.

Finite dimensional factor $\leftrightarrow$ complex matrix algebra

Three types of ∞-dim factors, classified by eq. classes of projections:

- Type I: $M = B(H)$ - no finite trace - projections classified by rank
- **Type II_1:** admits finite continuous trace - proj. classified by trace

  (**Type II_∞:** infinite matrices over type II_1)
- **Type III:** all projections equivalent

Examples of factors

Definition

The group von Neumann algebra LG of a discrete group is the closure of the group algebra in the left regular representaion on $L^2(G)$ (defined by $(gf)(h) = f(g^{-1}h)$ for $f \in L^2(G)$) in the topology of pointwise convergence.

The group von Neumann algebra is a factor of Type II_1 iff all the nontrivial conjugacy classes of G are infinite. The (unique) trace restricted to the group algebra gives the coefficient of the identity.

Example: Free groups $G = F_n, n \geq 2$.

$LF_2 \cong LF_3$? Open!

Powers factors

Let ϕ^0_λ, for $0 < \lambda < 1$, be the state of $M_2(\mathbb{C})$ defined by $\phi^0_\lambda(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \lambda a + (1-\lambda)d$. Consider $M^0 = \otimes_{n=0}^{\infty} M_2(\mathbb{C})$ with the product state ϕ_λ obtained by applying ϕ^0_λ to each factor. Let $L^2(M^0, \phi_\lambda)$ be the Hilbert space completion of M^0 with respect to the inner product induced by ϕ_λ. M^0 acts on $L^2(M^0, \phi_\lambda)$ by multiplication, continuously extended. Let $M = M^{0\prime\prime}$ (**GNS construction**.) M is a **hyperfinite** factor - limit of finite dimensional subalgebras. It is of Type II_1 if $\lambda = \frac{1}{2}$; Type III otherwise.

Hyperfinite II_1 factor unique

Classification of hyperfinite Type III factors - Connes

Subfactors

Definition

A subfactor is a unital inclusion $N \subset M$ of II_1 factors. It has **finite-index** if M is finitely generated as a module over N.

Example: Let G be a finite group acting by outer automorphisms on M. Then $M^G \subset M^H$ is a finite-index subfactor.

Galois correspondence: The intermediate subfactors $N \subset P \subset M$ are precisely the fixed point algebras M^K for $G \supset K \supset H$.

Every finite group has a unique (up to conjugacy) action on the hyperfinite II_1 factor; can recover group from subfactor.

Invariants for finite-index subfactors

- **Index** - number
- **Principal graphs** - pair of bipartite graphs
- **Fusion algebra** - tensor category
- **Standard invariant** - planar algebra

The basic construction

Let $N \subset M$ be a subfactor with Hilbert space completions $L^2(N) \subset L^2(M)$, and let $e_N : L^2(M) \twoheadrightarrow L^2(N)$ be the associated projection. Then $M_1 =< M, e_N >$ is a vN algebra on $L^2(M)$. The **Jones Index** is $[M : N] = tr(e_N)^{-1}$ if M_1 is finite or ∞ otherwise.

Can iterate:

$$M \subset M_1 \subset M_2$$

to obtain a **tower**:

$$N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 ...$$

with **Jones projections** $e_i : L^2(M_{k-1}) \twoheadrightarrow L^2(M_{k-2})$.

For groups:

$M^G \subset M \subset M \rtimes G$

The fusion algebra and principal graph

$N \subset M$ a subfactor, $[M : N] < \infty$, tower $N \subset M \subset M_1 \subset M_2 \ldots$.

Intertwiner Spaces

Let $\rho = {}_N M_M, \bar{\rho} = {}_M M_N$, $\rho^k = \overbrace{\rho \otimes_M \bar{\rho} \otimes_N \rho \ldots}^{k}$. Then $N' \cap M_{k-1} \cong End(\rho^k)$.

Can define a bipartite graph:

Even vertices: $E = \{$ isomorphism classes of irreducible $N - N$ bimodules occuring in the decomposition of ρ^k for some even $k\}$

Odd vertices: $O = \{$ isomorphism classes of irreducible $N - M$ bimodules occuring in the decomposition of ρ^k for some odd $k\}$

# of edges connecting $\alpha \in E$ to $\beta \in O$: multiplicity of β in $\alpha \otimes_N \rho$

Duality: Get another graph from $M \subset M_1$; however $M_1 \subset M_2$ gives the same graph as $N \subset M$.

For $M^G \subset M$, each graph has exactly one odd vertex.

One graph has an even vertex corresponding to each irreducible representation of G; bimodules tensor same way as group reps.

The other graph has an even vertex corresponding to each group element; bimodules tensor as group multiplication.

$End(\rho\bar{\rho}) \cong \mathbb{C}G$ and $End(\bar{\rho}\rho) \cong L^\infty(G)$

The standard invariant

$N \subset M$ a subfactor, $[M : N] < \infty$, tower $N \subset M \subset M_1 \subset M_2 \ldots.$

Definition

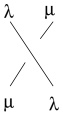
The **standard invariant** is

$$\begin{array}{ccccccc} (N' \cap N) & \subset & (N' \cap M) & \subset & (N' \cap M_1) & \subset & \ldots \\ & & \cup & & \cup & & \\ & & (M' \cap M) & \subset & (M' \cap M_1) & \subset & \ldots \end{array}$$

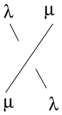
$N' \cap N = \mathbb{C}Id$ - always trivial
$N' \cap M$ may be trivial (**irreducible** subfactor)
$N' \cap M_1 \ni e_1$ - never trivial
...
$N' \cap M_k \supseteq \{e_1, \ldots, e_k\}$

Definition

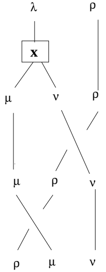
A **braiding** for a subfactor is a choice of bimodules $\{\lambda_i\}_{i \in E}$ representing the even vertices of the principal graph, and a set of unitaries $\varepsilon_{\lambda,\mu} : \lambda \otimes \mu \to \mu \otimes \lambda$ satisfying the **braiding fusion equations**. A braiding is **nondegenerate** if the only bimodule λ for which $\varepsilon_{\lambda,\mu} = \varepsilon^*_{\mu,\lambda}$ for all μ is the identity bimodule ${}_N N_N$.



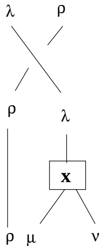
$:= \varepsilon_{\lambda,\mu}$,



$= \varepsilon^*_{\mu,\lambda}$, B.F.E.:

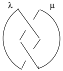


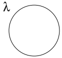
$=$



The modular group of the torus is generated by the transformations $\tau \mapsto 1+\tau$ and $\tau \mapsto -\frac{1}{\tau}$. These can be represented by the matrices $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ acting as fractional linear transformations.

If $N \subset M$ is a braided subfactor, then the matrix given by the traces of the braiding operators $\varepsilon(\lambda,\mu)\varepsilon(\mu,\lambda)$ is the image of S in a representation of the double cover of the modular group for which the image of T is diagonal.(Rehren, Turaev).

S: 

, T: 

Modular invariants (nonnegative integer matrices which commute with S and T) studied (Ocneanu, Xu, Bockenhauer, Evans, Kawahigashi, Gannon).

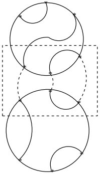
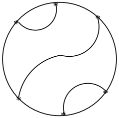
The Temperley-Lieb algebras

TL relations

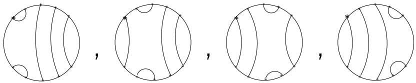
The Jones projections of a subfactor $e_1, e_2, \ldots$ satisfy:

$$e_i e_{i\pm 1} e_i = \tau e_i, \quad e_i e_j = e_j e_i \quad |i-j| > 1$$

Kaufmann diagrammatics: Let $TL_n(\delta)$ be the complex algebra defined on a basis of planar diagrams on $2n$ points with multiplication given by "concatenation of diagrams", where each closed loop contributes a factor of δ.

Example:  $= \delta$  (in TL_3)

A set of generators

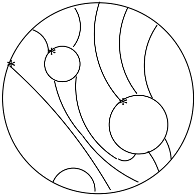
E_1, E_2, E_3, E_4: 

Relations: $E_iE_j = E_jE_i$ if $|i-j| > 1$, $E_iE_{i\pm1}E_i = E_i$.

Let $N \subset M \subset^{e_1} M_1 \subset^{e_2} M_2 \ldots$ be the Jones tower of a subfactor. There is a surjective map $TL([M:N]^{\frac{1}{2}}) \twoheadrightarrow e_1, e_2, \ldots$ given by $E_i \mapsto \delta e_i$. If $[M:N] \geq 4$ then this map is an isomorphism.

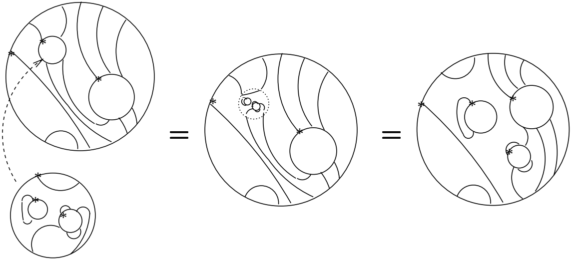
Planar tangles

Example:



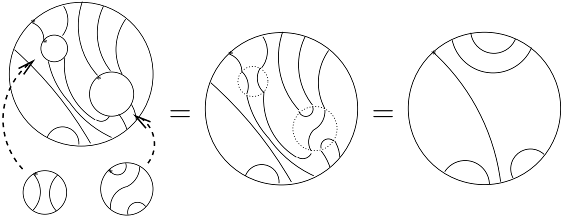
Think of internal discs an "inputs" and external disc as "outputs".
Tangles can be composed:

Example:



Planar Algebras

A **planar tangle** acts on the Temperley-Lieb algebras by gluing the Temperley-Lieb diagrams into the input discs of the tangle and removing the boundary.



Here we think of the tangle as a map from $TL_2 x TL_3$ to TL_5.

Definition

A **planar algebra** consists of a vector space

$$P_0 \subset P_1 \subset P_2 \subset \ldots$$

along with an action of the operad of planar tangles.

The standard invariant of a subfactor has a planar algebra structure taking $P_k = N' \cap M_{k-1}$ (Jones).

Jones projections $\Longleftrightarrow$ **TL algebras**
standard invariant $\Longleftrightarrow$ **planar algebra**

A **subfactor planar algebra** is a planar algebra with some conditions on maps to P_0 - necessarily arises as standard invariant of a subfactor (Popa).

Original proof very difficult and technical - what is the underlying subfactor?

New proof using planar algebras - Guionnet-Jones-Shlyakhtenko

Underlying subfactor is described in terms of planar algebra. Random matrix techniques used in proof.

What kind of factors involved?

Free group factor on infinitely many generators universal for subfactor theory - Popa-Shlyakhtenko

Subfactors with index less than 4

Jones Index Theorem: Let $N \subset M$ be a subfactor with $[M : N] < 4$. Then $[M : N] = 4cos^2 \frac{\pi}{k}$ for some $k = 3, 4, \ldots.$

Subfactors with $[M : N] < 4$ are classified up to isomorphism of the planar algebra by their principal graphs, which are Coxeter-Dynkin diagrams of Type A_n, D_{2n}, E_6, E_8, except that there are two different ones each for E_6 and E_8 (Ocneanu)..

Every index above 4 is realized - but by irreducible, hyperfinite subfactors? Still unknown.

Subfactors with index 4 have principal graphs

$$A_\infty, A_{-\infty,\infty}, D_\infty, A_n^{(1)}, D_n^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$$

A subfactor has **finite depth** if the principal graph is finite. List of possible graphs of small index finite depth subfactors - Haagerup

Smallest index above 4: **Haagerup subfactor** with index $\frac{5+\sqrt{13}}{2}$

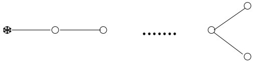
Also interesting: **Haagerup-Asaeda subfactor** with index $\frac{5+\sqrt{17}}{2}$

Extended Haagerup subfactor recently constructed using planar algebra techniques (Bigelow-Morrison-Peters-Snyder).

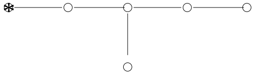
Principal graphs and supertransitivity

A subfactor is **k-supertransitive** if the first k levels of the planar algebras are generated by "empty" diagrams.

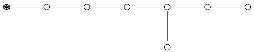
A_n is k-supertransitive for all k.



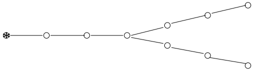
D_{2n} is $2n-3$-supertransitive.



E_6 is 2-supertransitive.



E_8 is 4-supertransitive.



the **Haagerup subfactor** is 3-supertransitive.

Quadrilateral of factors:

$$\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ N & \subset & Q \end{array} \quad \text{where } P \vee Q = M, P \wedge Q = N.$$

Dual quadrilateral:

$$\begin{array}{ccc} P' & \subset & N' \\ \cup & & \cup \\ M' & \subset & Q' \end{array}$$

A quadrilateral **commutes** if $e_P e_Q = e_Q e_P$. It **cocommutes** if its dual commutes.

Sano and Watatani: **angles**

$$Ang(P, Q) = spec(cos^{-1}(\sqrt{e_P e_Q e_P - (e_P \wedge e_Q)}))$$

Quadrilaterals with no extra structure

Commuting + ccocommuting: tensor product.

Noncommuting quadrilaterals?

Example: Let $G = S_3$, outer action on M, H and K distinct order 2 subgroups. Then

$$\begin{array}{ccc} M^H & \subset & M \\ \cup & & \cup \\ M^G & \subset & M^K \end{array}$$

does not commute (since $HK \neq KH$), but does cocommute. The elementary subfactors are all supertransitive.

Theorem (G-Jones)

Let $\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ N & \subset & Q \end{array}$ *be a noncommuting quadrilateral such that the elementary subfactors are supertransitive. Then either*

1) N is the fixed-point algebra of an outer action of the symmetric group S_3 on M

or

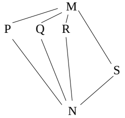
2) $[M : P] = [P : N] = 2 + \sqrt{2}$.

In either case the planar algebra for $N \subset M$ is uniquely determined.

True even if only 4-supertransitive (Izumi).

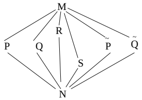
The two examples

The S_3 quadrilateral is cocommuting and we have $[M : P] = [M : Q] = 2$ and $[P : N] = [Q : N] = 3$. The full intermediate subfactor lattice is



and the angle between P and Q is $\pi/3$.

The other quadrilateral is noncocommuting, all of the sides have index $2 + \sqrt{2}$, the full intermediate subfactor lattice is



and the angle between P and Q is $\cos^{-1}(\sqrt{2} - 1)$.

Theorem (G-Izumi)

Let $\begin{array}{ccc} P & \subset & M \\ \cup & & \cup \\ N & \subset & Q \end{array}$ *be a noncommuting quadrilateral such that the elementary subfactors are 3-supertransitive. Then either*

1) the quadrilateral cocommutes and $[M : P] = [P : N] - 1$

or

2) the quadrilateral does not cocommute and $[M : P] = [P : N]$*.*

Small-index quadrilaterals

Theorem (G-Izumi)

Let $\begin{matrix} P & \subset & M \\ \cup & & \cup \\ N & \subset & Q \end{matrix}$ *be a noncommuting quadrilateral with* $[M:P], [M:Q], [P:N], [Q:N] \leq 4$. *Then the principal graphs* $(G_{N\subset P}, G_{P\subset M}) = (G_{N\subset Q}, G_{Q\subset M})$ *are one of the following pairs:*

$$(A_7, A_7), \quad (E_7^{(1)}, E_7^{(1)})$$

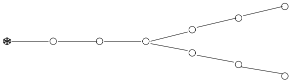
$$(A_5, A_3), \quad (D_6, A_4), \quad (E_7^{(1)}, A_5), \quad (E_6^{(1)}, D_4)$$

$$(D_6^{(1)}, A_3)$$

There is a unique planar algebra with each configuration.

Quadrilaterals of Haagerup subfactors

Haagerup subfactor - index $\frac{5+\sqrt{13}}{2}$, principal graph



1)Exists noncommuting, noncocommuting quadrilateral such that all of the elementary subfactors have index $\frac{5+\sqrt{13}}{2}$.

2)Exists noncommuting but cocommuting quadrilateral such that the upper subfactors have index $\frac{5+\sqrt{13}}{2}$ and the lower subfactors have index $\frac{7+\sqrt{13}}{2}$.