

Compactness of Schrödinger semigroups with unbounded below potentials

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In the study of Schrödinger operators

$$H := -\Delta + V$$

with $V \in L^1_{loc}(\mathbb{R}^d)$ being nonnegative, one fundamental question is under which criteria, H has purely discrete spectrum, i.e.,

$$\sigma_{\text{ess}}(H) = \emptyset.$$

One celebrated condition, due to K. Friedrichs (*Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren*, **Math. Ann.** **1934**), is $\lim_{|x| \rightarrow \infty} V(x) = \infty$.

Moreover, cf. e.g. Reed and Simon: *Methods of Modern Mathematical Physics IV: Analysis of Operators* (Theorem XIII),

$$\sigma_{\text{ess}}(H) = \emptyset \iff e^{-tH} \text{ is compact } \forall t > 0.$$

Thus, *discreteness of the spectrum of H is equivalent to the compactness of the corresponding Schrödinger semigroup $\{e^{-tH}\}_{t>0}$* . In particular, in case that $V \geq 0$, a Feynman-Kac semigroup generated by $-\Delta + V$ on \mathbb{R}^d is compact provided $\lim_{|x| \rightarrow \infty} V(x) = \infty$. This then becomes a usual and classical criterion in the study of the compactness for Schrödinger type semigroups.

In this spirit, T. Kulczycki and B. Siudeja (**Trans. Amer. Math. Soc. 2006**) studied intrinsic ultracontractivity of the Feynman-Kac semigroups for relativistic α -stable processes.

The usual and classical condition attracts renewed considerations in V. Maz'ya and M. Shubin (*Discreteness of the spectrum and positivity criteria for Schrödinger operators*, **Ann. Math.** **162** (2005), 919-942).

Our object here is to formulate a better criterion for the compactness of Schrödinger semigroups in an abstract framework so that it can apply to a broad class of potentials unbounded below on a set of finite volume. That is, we aim to prove the L^2 -compactness of a class of Schrödinger semigroups with unbounded below potentials.

Our main tool is to utilise Wang's intrinsic super Poincaré inequality initiated by F.-Y. Wang (**J. Funct. Anal.** **2002**)

According to *Decomposition Principle* – the essential spectrum is invariant under perturbation of compact operators, H. Donnelly and P. Li (**Duke Math. J. 1979**) gave the following principle for the Laplacian on a complete Riemannian manifold M :

$$\inf \sigma_{\text{ess}}(-\Delta) \geq \sup_{K \subset M \text{ is compact}} \inf \left\{ \frac{\int_M |\nabla f(x)|^2 \lambda(dx)}{\int_M f(x)^2 (\lambda(dx))} : f \in C_0^1(M), f|_K = 0 \right\}.$$

F.-Y. Wang (**J. Funct. Anal. 2000**) verified that the equality holds and extended the formula to a general situation as long as there holds a Sobolev embedding theorem. This is our starting point.

Let E be a Polish space with Borel σ -algebra \mathcal{F} , and μ a σ -finite measure on E . Let $(\mathcal{E}_0, \mathcal{D}(\mathcal{E}_0))$ be a symmetric Dirichlet form on $(L^2(\mu), \langle \cdot, \cdot \rangle)$ with generator $(L_0, \mathcal{D}(L_0))$ [note: a self-adjoint operator on $(L^2(\mu), \langle \cdot, \cdot \rangle)$]

$$\mathcal{E}_0(f, g) := - \langle f, L_0 g \rangle .$$

Let V be a measurable function on E s.t. $\mathcal{D}(L_0) \cap L^2(|V|\mu)$ is dense in $L^2(\mu)$. If the operator $L_V := L_0 - V$ is bounded above in the sense of quadric form, i.e.

$$\begin{aligned} \mathcal{E}_V(f, f) &:= \mathcal{E}_0(f, f) + \mu(Vf^2) \geq -c(V)\mu(f^2) \\ f &\in \mathcal{D}(L_0) \cap L^2(|V|\mu) \end{aligned}$$

holds for some constant $c(V) \geq 0$ (where $\mu(h) := \int_E h d\mu$ for $h \in L^1(\mu)$), then $(L_V, \mathcal{D}(L_V))$, the Friedrichs extension of $(L_V, \mathcal{D}(L_0) \cap L^2(|V|\mu))$, generates a bounded semigroup P_t^V on $L^2(\mu)$.

Furthermore, P_t^V is positivity preserving (cf. the proof of Theorem 1.8.2 of E.B. Davies' monograph: *Heat Kernels and Spectral Theory* (CUP, 1989)).

In order to establish functional inequalities for the Schrödinger operator, we use the nonnegative quadric form $\tilde{\mathcal{E}}_V := \mathcal{E}_V + c(V)$ in stead of \mathcal{E}_V . Obviously, the compactness of P_t^V is equivalent to that of the contractive semigroup associated to \mathcal{E}_V .

To study the compactness of P_t^V , we first consider the semicompactness of P_t^V . For a bounded linear operator P on $L^2(\mu)$, the tail norm

$$\|P\|_T := \lim_{K \rightarrow \infty} \sup_{\mu(f^2) \leq 1} \mu((Pf)^2 \mathbf{1}_{\{|Pf| \geq K\varphi\}})$$

is independent of the choice on strictly positive functions $\varphi \in L^2(\mu)$, and is called the measure of non-semicompactness of P . If $\|P\|_T = 0$ then the operator P is called semicompact on $L^2(\mu)$.

According to Theorem 3.2.2 of F.-Y. Wang's monograph: *Functional Inequalities, Markov Semigroups and Spectral Theory* (Science Press, 2005), P_t^V is semicompact for any $t > 0$ if and only if there exists a positive function $\varphi \in L^2(\mu)$ such that

$$\mu(f^2) \leq r\mathcal{E}_V(f, f) + \beta(r)\mu(|f|\varphi)^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}_V)$$

holds for some (decreasing) function $\beta : (0, \infty) \rightarrow (0, \infty)$. This inequality is called *the intrinsic super Poincaré inequality* and has been applied to the study of intrinsic ultracontractivity (cf. E.M. Ouhabaz and F.-Y. Wang, **Manuscripta Math.**, 2007).

Next, due to Theorem 3.1.7 of Wang's monograph, a bounded operator is compact if and only if it is semicompact and has an asymptotic kernel. Recall that a bounded linear operator P on $L^2(\mu)$ is called a kernel operator if there exists a measurable function ϱ on $E \times E$ such that for any $f \in L^2(\mu)$ one has $\int_E |\varrho(\cdot, y)f(y)|\mu(dy) \in L^2(\mu)$ and

$$Pf = \int_E \varrho(\cdot, y)f(y)\mu(dy).$$

In this case ϱ is called the density of P with respect to μ .

Moreover, P is said to have an asymptotic kernel if

$\|P - P_n\|_2 \rightarrow 0$ as $n \rightarrow \infty$ for a sequence of kernel operators $\{P_n\}$.

So, to prove the compactness of P_t^V , one only has to verify the intrinsic super Poincaré inequality and the existence of density of P_t^V with respect to μ . In most situations for finite-dimensional models, the latter can be observed easily. So, the main point of the study reduces to the validity of the intrinsic super Poincaré inequality. To this end, we start from the super Poincaré inequality for \mathcal{E}_0 which appears naturally in applications:

$$\mu(f^2) \leq r\mathcal{E}_0(f, f) + \beta_0(r)\mu(|f|)^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}_0)$$

for some $\beta_0 : (0, \infty) \rightarrow (0, \infty)$. For instance, for the Brownian motion or the α -stable process on \mathbb{R}^d there holds a Nash inequality, which is equivalent to super Poincaré inequality with $\beta_0(r) = cr^{-d/2}$ or $cr^{-d/\alpha}$ for some $c > 0$. (cf. A. Bendikov and P. Matheaux: *Nash type inequalities for fractional powers of non-negative self-adjoint operators*, **Trans. Amer. Math. Soc.** 2007.)

Theorem

Let the super Poincaré inequality hold for some decreasing β_0 and let V satisfy

$$\mu((K + V)^{-}\beta_0(\delta/(K + V)^{-})) < \infty, \quad \delta > 0$$

for some constant $K > 0$. Then the Friedrichs extension L_V is bounded above and hence the associated semigroup P_t^V is a well-defined bounded semigroup on $L^2(\mu)$.

Furthermore, if

$$\mu(V < N) < \infty$$

for any $N > 0$, then P_t^V exists and is semicompact on $L^2(\mu)$ for any $t > 0$. Consequently, if moreover P_t^V has a density with respect to μ , then it is compact on $L^2(\mu)$.

If the super Poincaré inequality holds for some β_0 satisfying

$$\int_s^\infty \frac{\beta_0^{-1}(r)}{r} dr < \infty$$

for large $s > 0$, where $\beta_0^{-1}(r) := \inf\{t > 0 : \beta(t) \leq r\}$, then $P_t^0 := e^{tL_0}$ is ultracontractive so that the semigroup P_t^0 has a bounded density with respect to μ for any $t > 0$ (cf. Wang's monograph). Indeed, one can prove the same assertion for P_t^V provided our above condition

$$\mu((K + V)^- \beta_0(\delta / (K + V)^-)) < \infty, \quad \delta > 0$$

holds.

Corollary

Assume all above hold, then P_t^V is compact on $L^2(\mu)$ for all $t > 0$.

Remark Although Condition

$$\int_s^\infty \frac{\beta_0^{-1}(r)}{r} dr < \infty$$

is not a strong assumption for infinite reference measures due to Nash inequalities on \mathbb{R}^d or on a Riemannian manifold, it is quite restrictive for finite μ as it implies the ultracontractivity of P_t^0 , so that even the standard Ornstein-Uhlenbeck semigroup is excluded. In order to provide a reasonable alternative to this condition in the finite measure setting, we could make use of the Feynman-Kac formula ensured by

$$\mathbf{E}^x e^{\int_0^t V^-(X_s) dx} < \infty, \quad x \in E,$$

where X_s is the right-continuous strong Markov process generated by L_0 .

Corollary

Assume that L_0 generates a unique right-continuous strong Markov process whose semigroup having a transition density with respect to μ , and the super Poincaré inequality holds for some decreasing β_0 . Moreover, let V satisfy

$$\mu((K + V)^{-\beta_0}(\delta/(K + V)^{-})) < \infty, \quad \delta > 0$$

and

$$\mathbf{E}^x e^{\int_0^t V^-(X_s) dx} < \infty, \quad x \in E.$$

Then P_t^V is compact on $L^2(\mu)$ for $t > 0$ provided $\mu(V < N) < \infty$ for all $N > 0$.

To apply these results to subordinated Schrödinger semigroups, recall that a smooth nonnegative function g on $[0, \infty)$ is called *Bernstein function*, if $(-1)^n g^{(n)} \leq 0$ for all $n \geq 1$. It is well-known that $-g(-L_0)$ is a Dirichlet operator for any Bernstein function g (cf. e.g. C. Berg and G. Forst: *Potential Theory on Locally Compact Abelian Groups*. Springer-Verlag, 1975).

Proposition

(Super Poincaré inequality by subordination) Let g be a Bernstein function with $g(r) \rightarrow \infty$ as $r \rightarrow \infty$, and let $\mathcal{E}^{(g)}$ the Dirichlet form associated to $-g(-L_0)$. If the super Poincaré inequality holds for \mathcal{E}_0 , then there exists $\beta_g : (0, \infty) \rightarrow (0, \infty)$ such that

$$\mu(f^2) \leq r\mathcal{E}^{(g)}(f, f) + \beta_g(r)\mu(|f|)^2, \quad r > 0, f \in \mathcal{D}(\mathcal{E}^{(g)}).$$

That is, the super Poincaré inequality is stable under subordination! This together with our theorem implies

Corollary

Let the super Poincaré inequality hold for some β_0 and assume that P_t^0 has a density with respect to μ . Then for any Bernstein function g with $g(r) \uparrow \infty$ as $r \uparrow \infty$, and for any bounded below measurable function V on E with $\mu(V < N) < \infty$ for all $N > 0$, the semigroup generated by $-g(-L_0) - V$ is compact in $L^2(\mu)$.

The first example is a generalization to Schrödinger semigroups for relativistic α -stable process studied in T. Kulczycki and B. Siudeja (**Trans. Amer. Math. Soc.** 2006). Due to the Nash inequality, the super Poincaré inequality holds for $L_0 = \Delta$ on \mathbb{R}^d with $\beta_0(r) = cr^{d/2}$ for some $c > 0$. Thus, this example considerably strengthens Theorem 1.1 there, due to the fact that the present potential V is allowed to be unbounded below and not necessarily going to infinity at infinity.

Example 1 Let the super Poincaré inequality hold for $\beta_0(r) = c(1 + r^{-\delta})$ with some $c, \delta > 0$, and let $\alpha \in (0, 2]$. If $\mu((V^-)^{1+2\delta/\alpha}) < \infty$ then the Schrödinger operator

$$L_{V,m}^{(\alpha)} := -(-L_0 + m^{2/\alpha})^{\alpha/2} + m - V$$

generates a bounded semigroup $P_t^{V,m,\alpha}$ on $L^2(\mu)$, which is compact if furthermore $\mu(V < N) < \infty$ for any $N > 0$.

The next example considers Schrödinger semigroups for a class of typical diffusion processes with probability invariant measures, for which the potential V is even allowed to go to minus infinity algebraically as the distance going to infinity.

Example 2 Let E be a connected complete Riemannian manifold with Ricci curvature bounded below, and let ρ be the Riemannian distance to a fixed point. Let $\mu(dx) = e^{-\rho^\delta} dx$ for some $\delta > 1$, where dx is the volume measure on E . Let $\mathcal{E}_0(f, g) = \mu(\langle \nabla f, \nabla g \rangle)$ with $\mathcal{D}(\mathcal{E}_0) := W^{1,2}(\mu)$. Finally, let $V \geq -\lambda\rho^\theta$ for some $\lambda, \theta > 0$. If $\theta < 2(\delta - 1)$, then P_t^V is a well-defined, compact semigroup on $L^2(\mu)$.

The key step in the proof here is to use the Laplacian comparison theorem and Kendall's Itô formula for the radial process (cf. W.S. Kendall: The radial part of Brownian motion on a manifold: a semimartingale property. **Ann Probab**, 1987).

Example 3 Given a countable set \mathbf{S} . For any σ -finite measure μ on \mathbf{S} with full support and for any nonnegative measurable function V such that $\mu(V \leq N) = \infty$ for some $N > 0$, set $E_0 := \{V > N\}$. One can then construct a conservative Markov chain (birth-death process) on E_0 whose Dirichlet form is symmetric in $L^2(E_0, \mu)$ and satisfies the super Poincaré inequality, but P_t^V can not be semi-compact for any $t > 0$!

This example shows that the condition $\mu(V \leq N) < \infty$ for all $N > 0$ is essential for the compactness of P_t^V in the context of countable Markov chains. It also provides a new kind example of Markov chains, e.g., to Mufa Chen's classic book: *From Markov Chains to Non-Equilibrium Particle Systems*, World Scientific, 1992.

[1] B. Simon, Schrödinger operators with purely discrete spectrum, *Methods of Functional Analysis and Topology*, 15 (2009), 61–66.

gives an “elementary” proof of our theorem (using results from Reed and Simon’s Volumes I and IV) and various extensions.

[2] D. Lenz, P. Stollmann and D. Wingert, Compactness of Schrödinger semigroups, *Mathematische Nachrichten* (2010).

presents an alternative proof of our theorem and extends to operator theoretical settings.

[3] V. Georgescu, Hamiltonians with purely discrete spectrum, <http://arxiv.org.abs/0810.5563vl>.

links our theorem to self-adjoint operators on a locally abelian groups.

[4] F.-Y. Wang and J.-L. Wu, Intrinsic ultracontractivity of the Schrödinger semigroups with unbounded below potentials, in preparation.

The notion of intrinsic ultracontractivity was introduced by E.B. Davies and B. Simon (*Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians*, **J. Funct. Anal.** **59** (1984), 335 – 395) in a general setting of submarkovian semigroups, in order to study the semigroups of killed Brownian motion and Schrödinger operators. Since then the problem has been studied by many authors for various type of processes.

Let us mention a few works related to ours:

1. R. Carmona, W. Masters and B. Simon *J. Funct. Anal.* 1990
2. T. Kulczycki and B. Siudeja *Trans. Amer. Math. Soc.* 2006
3. E.M. Ouhabaz and F.-Y. Wang *Manuscripta Math.* 2007
4. M. Kwasnicki *Potential Anal.* 2009.

This problem is closely related to estimates on the eigenfunctions of the Schrödinger type Markov generators, and to the parabolic (backward) boundary Harnack inequality (cf. e.g. E.B. Fabes, N. Garofalo and S. Salsa: *A backward Harnack inequality and Fatou theorem for nonnegative solutions of parabolic equations*, **Illinois J. Math.** 30 (1986), 536 – 565). Our investigation starts from this point and we will work out the Harnack inequality first.