Random planar curves
Schramm-Loewner Evolution
and Conformal Field Theory

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Introduction - lattice models in two dimensions and random planar curves

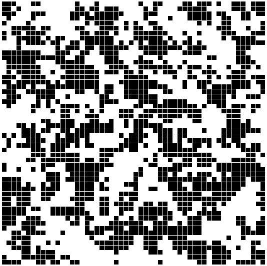
The Ising model

- configurations $\{s(r) = \pm 1\}$ with $r \in \delta^2\mathbb{Z}^2 \cap D$
- weighted by

$$\exp\left(\sum_{\text{edges } rr'} \delta\big(s(r), s(r')\big)/T\right)$$

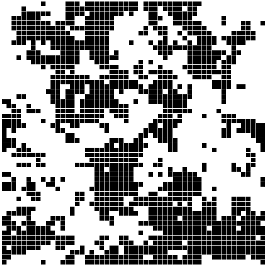
- defines a discrete probability measure on configurations

- typical configurations:



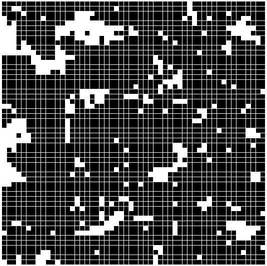
162x162, T>Tc

$T > T_c$



162x162, T=Tc

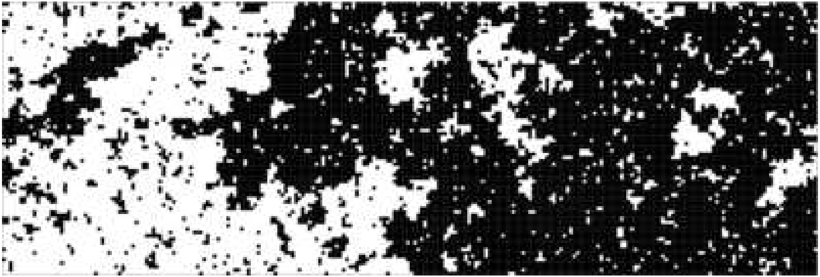
$T = T_c$



162x162 Lattice, T<Tc

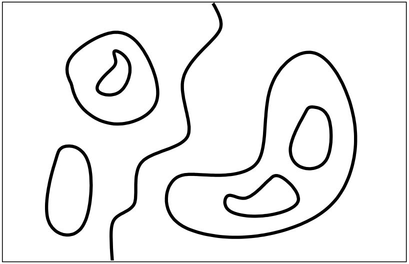
$T < T_c$

- a larger domain



$T = T_c$, $-/+$ boundary conditions

- cartoon version



- nested non-intersecting curves, weighted by $e^{-\text{total length}/T}$
- generalisation: weight each configuration by

$$e^{-\text{total length}/T}\, n^{\text{number of loops}}$$

- we are interested in the *scaling limit*: lattice spacing $\delta \to 0$ with D fixed, where many properties are supposed to be *universal*: different values of n have different scaling limits (*universality classes*)

Main approaches

- Integrability: find lattice models which are 'exactly solvable': however usually only extensive thermodynamic properties are calculable, and it is unclear which of these are universal
- Conformal Field Theory: directly describes the scaling limits of local correlation functions, eg

$$\mathbb{E}[s(r_1)s(r_2)] - \mathbb{E}[s(r_1)]\,\mathbb{E}[s(r_2)]\,;$$
$$\Pr\big(r_1 \text{ and } r_2 \text{ lie within } \epsilon \text{ of the same curve}\big)$$

  assuming they satisfy certain postulates, in particular *conformal covariance*
- Schramm-Loewner Evolution: directly describes the scaling limit of the measure on the planar curves, assuming fewer postulates, in particular *conformal invariance*

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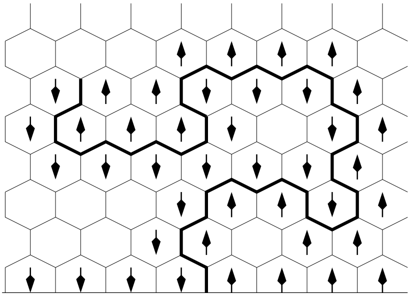
- these three approaches appear to be mathematically linked through the existence of holomorphic observables

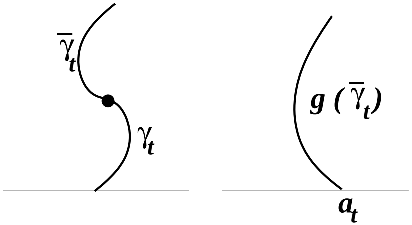
SLE

- choose D = upper half-plane $\mathbb{H}$, and boundary conditions such that there is a curve γ from 0 to ∞
- in the lattice this can be constructed via an *exploration process*



- let γ_t be the curve up to 'time' t, and $\overline{\gamma}_t = \gamma \setminus \gamma_t$
- conditional law of $\overline{\gamma}_t$ given γ_t in $\mathbb{H}$ = law of $\overline{\gamma}_t$ in $\mathbb{H} \setminus \gamma_t$
- SLE assumes that this 'domain Markov property' extends to the measure on the curve in the scaling limit

Loewner's Equation



- consider the conformal mapping $g_t : \mathbb{H} \setminus \gamma_t \to \mathbb{H}$ such that

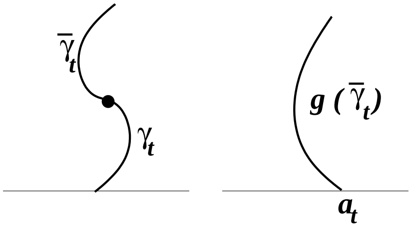
$$g_t(z) \stackrel{z\to\infty}{=} z + \frac{2t}{z} + o(z^{-1})$$

so that $g_t(\text{growing tip}) = a_t$. Then

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - a_t} \qquad \text{(Loewner)}$$

- if $\{\gamma_t\}$ is an increasing set of (random) curves which grow only at the tip then a_t is a continuous (random) process on the real line

Conformal Invariance of the Measure



- let $\gamma' = g_t(\overline{\gamma}_t) - a_t$, a curve from 0 to ∞
- if the measure is *conformally invariant* , $\mu(\gamma') = \mu(\gamma)$

Theorem (Schramm). *If both the domain Markov property and conformal invariance hold then a_t is proportional to standard Brownian motion: $a_t = \sqrt{\kappa} B_t$*

- different values of κ correspond to different universality classes: conjecturally $n = -2\cos(4\pi/\kappa)$
- identifying suitable martingales of this process allows the computation of many physically interesting quantities, e.g. fractal dimension of γ: $d_f = 1 + \frac{1}{8}\kappa$

CFT

- CFT is a special case of a QFT: a collection of functions $D^N \setminus \{\text{coincident points}\} \to \mathbb{C}$, denoted by $\langle \phi_1(r_1) \dots \phi_N(r_N) \rangle_D$
- they satisfy certain axioms, *e.g.* the short-distance expansion

$$\phi_i(r_i) \cdot \phi_j(r_j) = \sum_k C_{ijk}(r_i - r_k)\phi_k\big(\tfrac{1}{2}(r_i + r_j)\big)$$

- the conjectured scaling limit of correlation functions of lattice models gives an example of a QFT:

$$\lim_{\delta \to 0} \delta^{-x_1 \dots - x_N} \mathbb{E}\big[\phi_1^{\text{lat}}(r_1) \dots \phi_N^{\text{lat}}(r_N)\big] = \langle \phi_1(r_1) \dots \phi_N(r_N) \rangle$$

- at $T = T_c$ the QFT is *massless* and the absence of any intrinsic scale implies *scale covariance*

$$\langle \phi_1(r_1) \dots \phi_N(r_N) \rangle_D = \lambda^{x_1 \dots + x_N} \langle \phi_1(\lambda r_1) \dots \phi_N(\lambda r_N) \rangle_{\lambda D}$$

written more compactly as

$$\phi_j(r) = \lambda^{x_j} \phi_j(\lambda r)$$

- in a CFT this gets promoted to *conformal covariance*: under a conformal mapping $f : z \to z'$

$$\phi_j(z,\bar z) = f'(z)^{h_j} \left(f'(z_j)^*\right)^{\bar h_j} \phi_j(z',\bar z')$$

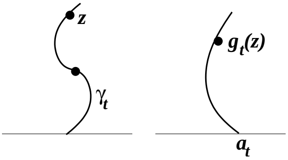
where $h_j + \bar h_j = x_j$ is the *dimension* and $h_j - \bar h_j = \sigma_j$ is the *conformal spin* of ϕ_j.

- a special role is played by *holomorphic* fields $\phi_\sigma(z)$ such that $(h,\bar h) = (\sigma, 0)$ (and similarly antiholomorphic fields)
- these have simple short-distance expansions with other fields:

$$\phi_\sigma(z)\cdot\Phi_j(0,0) = \sum_n z^{-\sigma-n}\left(W_n\Phi_j\right)(0,0)$$

- the W_n generate an infinite dimensional algebra $\mathcal{W}$ and the fields Φ_j of the CFT fall into highest weight representations of $\mathcal{W}\otimes\overline{\mathcal{W}}$
- this allows a classification and characterisation of CFTs

Holomorphic fields and SLE



- we can associate a field of conformal spin σ with the curve γ:

$$\phi_\sigma(z,\bar{z}) = \mathbf{1}_{\{z\in\gamma\}}\, e^{-i\sigma\theta_{0z}(\gamma)}$$

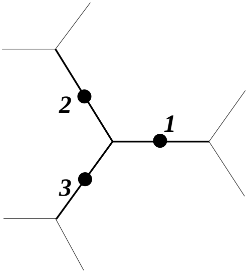
where $\theta_{0z}(\gamma)=$ winding angle of γ between 0 and z. Note that as $z\to\partial D$, $\arg\phi_\sigma$ is fixed. If ϕ_σ is holomorphic $\langle\phi_\sigma(z)\rangle_{\mathbb{H}}\propto z^{-\sigma}$, and

$$\begin{aligned}\langle\phi_\sigma(z)\rangle_{\mathbb{H}} &= \mathbb{E}_{\gamma_t}\left[\langle\phi_\sigma(z)\rangle_{\mathbb{H}\setminus\gamma_t}\right] = \mathbb{E}_{g_t}\left[g_t'(z)^\sigma\langle\phi_\sigma\big(g_t(z)\big)\rangle_{\mathbb{H}}\right]\\ &= \mathbb{E}_{a_t}\left[\frac{g_t'(z)^\sigma}{(g_t(z)-a_t)^\sigma}\right]\end{aligned}$$

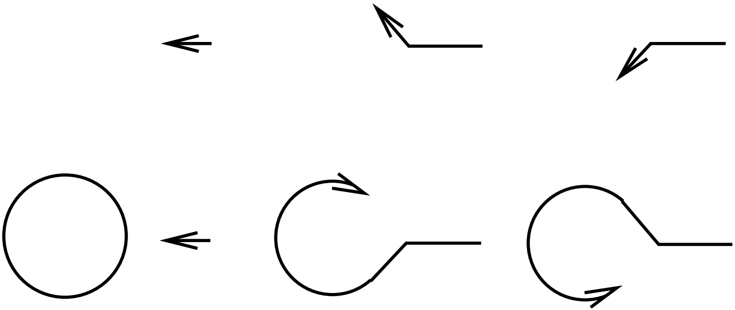
- $[\cdots]$ is a martingale: evaluating this for large z gives $\mathbb{E}[a_t]=0$ and $\mathbb{E}[a_t^2]=\kappa t$ where $\kappa=8/(\sigma+1)\Rightarrow\gamma$ is SLE

Discrete holomorphicity

- at the level of lattice models we can often show directly from the local weights that, for suitable σ, $\phi_\sigma(z)$ is *discretely holomorphic*, e.g.



$$\phi_\sigma(z_1) + \omega\phi_\sigma(z_2) + \omega^2\phi_\sigma(z_3) = 0$$



$$\phi_\sigma(z_1) + \omega\phi_\sigma(z_2) + \omega^2\phi_\sigma(z_3) = 0$$

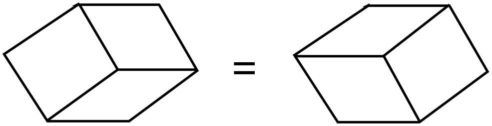
- however, it is a highly non-trivial step to show that this implies that, in the scaling limit $\delta \to 0$, $\phi_\sigma(z)$ converges to a truly holomorphic field, in fact only in one case has this been carried out completely:

Theorem (Smirnov): *For the Ising model on the square and triangular lattices $\phi_{\frac{1}{2}}(z)$ defined in this way converges to a holomorphic field obeying the correct boundary conditions, and hence the scaling limit of the lattice curve γ is SLE with $\kappa = 3$.*

Remark: in this case $\phi_{\frac{1}{2}}$ is related to the well-known Ising fermion in other exact solutions of the model.

Discrete Holomorphicity and Integrability

- by now, discretely holomorphic observables have been identified in many lattice models
- the values of σ correspond to those of the holomorphic parafermionic fields in the conjectured corresponding CFT
- discrete holomorphicity imposes linear relations between the local weights of the lattice models – in all known cases these turn out to be the critical subspace of the *integrable* manifold in the space of all possible weights - *i.e.* they satisfy the Yang-Baxter equations.



Summary

- Schramm-Loewner Evolution and Conformal Field Theory are complementary ways of describing the conjectured scaling limit of random curves in critical lattice models
- the inputs of SLE are fewer and more easily susceptible to verification - e.g. by proving the existence of suitable holomorphic observables
- in some cases it is possible to derive many of the fundamental equations of CFT from SLE (eg the conformal Ward identities)
- holomorphic observables play an important role in
  - showing that the scaling limit of lattice curves is SLE
  - identifying the holomorphic fields which are the building blocks of the corresponding CFT
  - identifying which critical lattice models are also integrable

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  - identifying the holomorphic fields which are the building blocks of the corresponding CFT
  - identifying which critical lattice models are also integrable
- many unresolved questions - why integrability? extension to Conformal Loop Ensemble (CLE)? . . .